A Memo:

Analytic Formulas For the Propagation of Photons in Ice from Cascades and Muons

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Analytic Formulas For the Propagation of Photons in Ice:

Why?

- PTD
- Photonics
- Neural net fits of the above.

But…

- Way to compare simulation to reality.
- Fast reconstruction algorithms vs mind boggling protocols… Table lookup time micro-seconds -> Years.
- Good fits to simulation (i.e. reality) are good because they are well motivated physically!
- Reduced memory load for quickly varying ice properties.
  - Table sizes depend not just upon OM spacing, but source positions.
  - (See talks by Bay, Japaridize&Ribordy).
- Not a Panacea, but inconsistencies between reconstructed data and MC may be not because of ice, but because PDFs improperly weight arrival times.

What do We *Want* to "Know" about Propagation for Doing Neutrino Astrophysics?

Want to be able to calculate

- the arrival time probability distributions, p(d,t),
- the total number of photoelectrons, n(d),
- the time of the peak of the arrival time distribution,
- the average time and width,

as functions of

- the direction of propagation of the incoming neutrino,
- the orientation of the OM relative to the source,
- the distance to the OM,

from:

A. A stationary point source of (non-isotropic) Cherenkov photons,

B. A moving point source of (non-isotropic) Cherenkov photons.

Because we need to be able to calculate:

- p(d,t) -> likelihoods,
- n(d) -> hit probability = $1 - e^{-n(d)}$, multi-pe statistics.

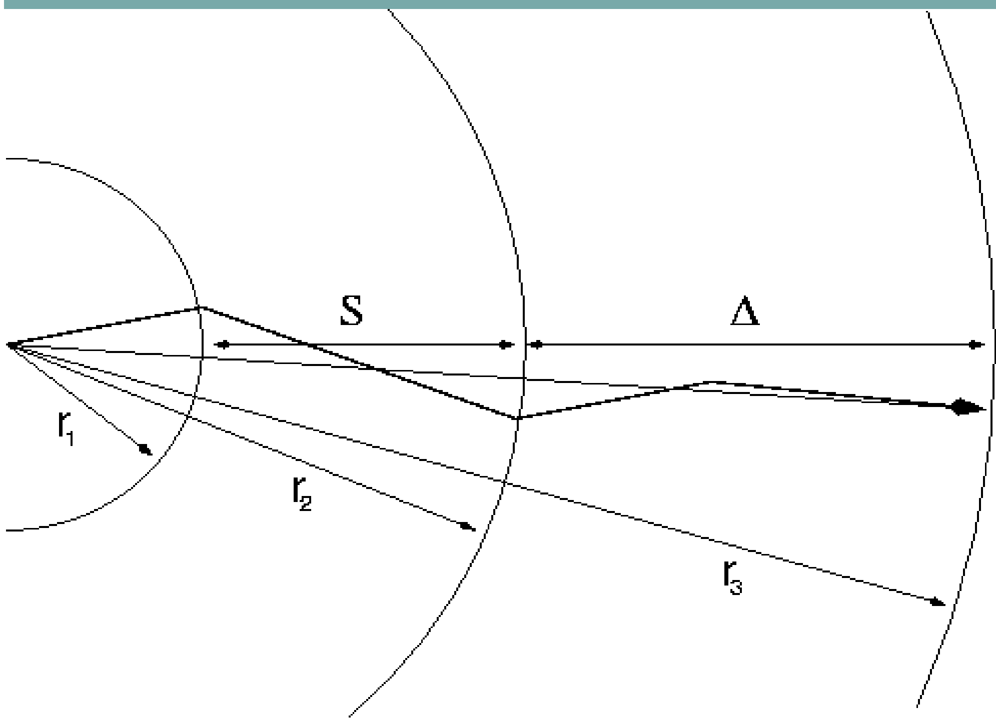
We have some parts of analytic solutions for *some situations*:

1. Stationary isotropic point sources:
   a) in the *symmetric* regime (Pandel's solution),
      - (temporal and spatial dependence)
   b) in the diffusive regime,
      - (temporal and spatial dependence)
   c) in the single scattering regime (Alcock and Hatchett)
      - (temporal, spatial and angular dependence).
   d) in the forward scattering regime (A.S. Monin)
      - (spatial and angular dependence).
2. Moving isotropic point sources:
   a) in the symmetric regime (recycled Pandel),
   b) in the diffusive regime (Porrata 1997)
      - (temporal and spatial dependence).

POINT SOURCES

(1)

Case (1a): Pandel's Considerations for the Symmetric Regime



- Perhaps in analogy to Feynman's sum over all paths…
- Plane or radial symmetry.
- Symmetry of no fixed reference, i.e., no definite starting point, ie, translational symmetry.
- Does not at all depend upon the exact details of the single particle scattering function.
- Is this *really* a scattering problem?

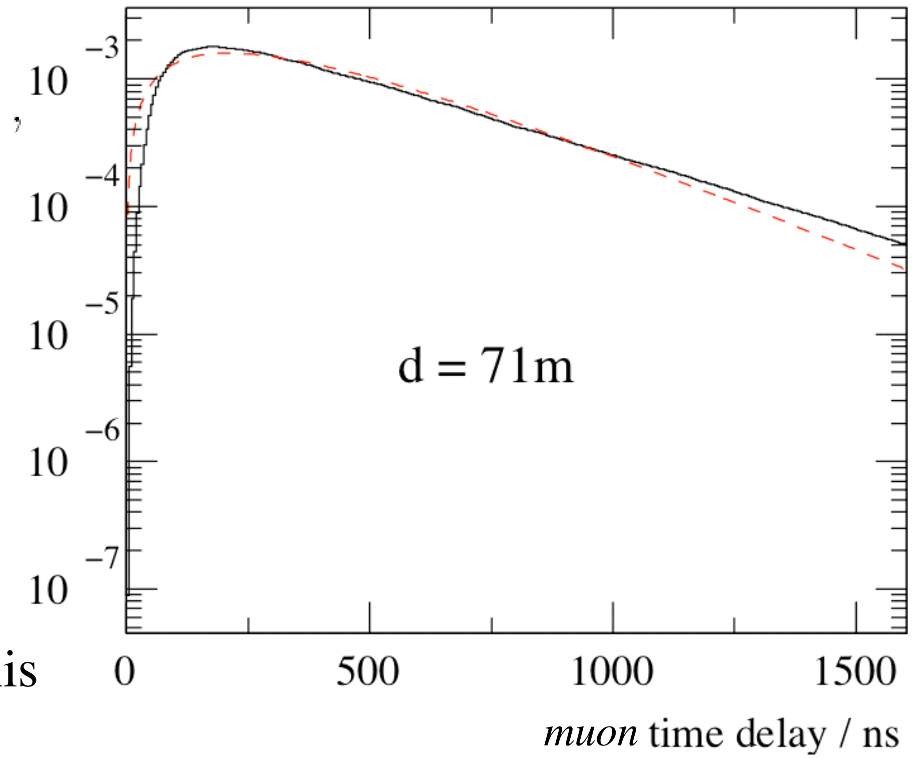
$$P(s+\Delta,t) = \int_0^t P(s,t')\ P(\Delta,t-t')\ dt'$$

D. Pandel, Diploma Thesis (1996).

Case (1a) : Pandel's Solution for the Symmetric Regime

$$p(t_{\text{res}}) \equiv \frac{1}{N(d)} \frac{\tau^{-(d/\lambda)} \cdot t_{\text{res}}^{(d/\lambda - 1)}}{\Gamma(d/\lambda)} \cdot e^{-\left(t_{\text{res}} \cdot \left(\frac{1}{\tau} + \frac{c_{\text{medium}}}{\lambda_a}\right) + \frac{d}{\lambda_a}\right)}$$

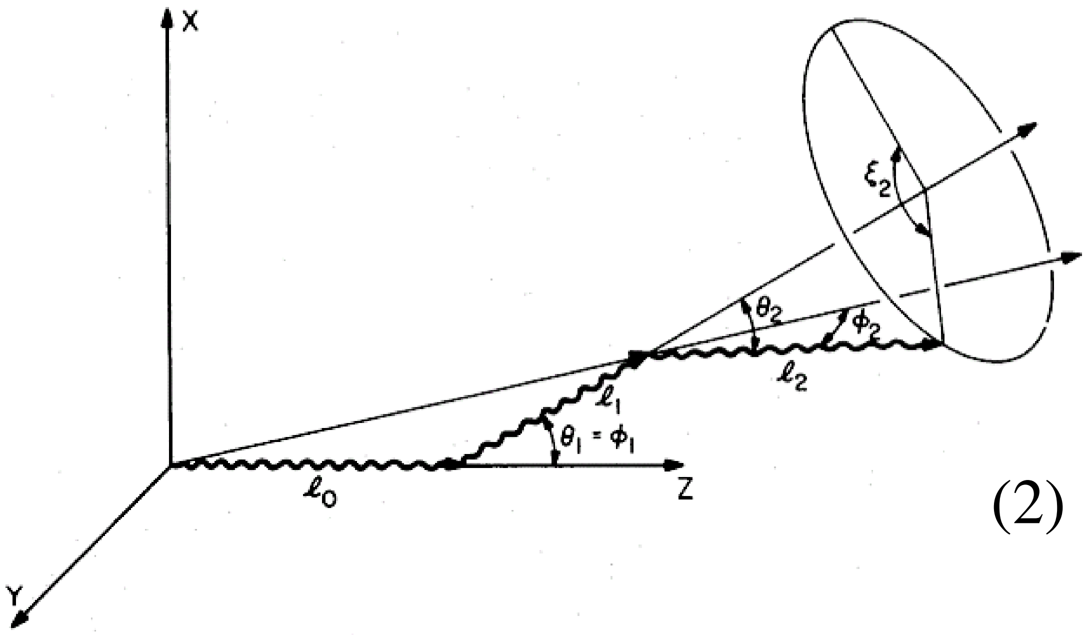
$$N(d) = e^{-d/\lambda_a} \cdot \left(1 + \frac{\tau \cdot c_{\text{medium}}}{\lambda_a}\right)^{-d/\lambda},$$



- Causal.
- Is NOT a Green function solution. It is a normalized PDF.
- *Meant* to be used for point sources. For this purpose the time distribution IS ok!
- Normalization does NOT give the right number of photons.
- Attempts to *analytically* integrate p(t) along muon track have so far failed.

D. Pandel, Diploma Thesis (1996).

Mean Squared Distance and Nth Scattering Green Function about Mean Distance



$$\langle \mathbf{r}^2 \rangle_N = \langle \left(\sum_{i=1}^{N} \mathbf{l}_i \right)^2 \rangle$$

$$\langle \mathbf{r}^2 \rangle_N = 2\lambda_s^2 \sum_{k=0}^{N-1} (N-k)\gamma^k$$

(2) $$\langle \mathbf{r}^2 \rangle_N = 2\lambda_e^2 [N\nu + \gamma(\gamma^N - 1)]$$

(3) $$W_N(r,s) = \left(\frac{3}{2\pi \langle \mathbf{r}^2 \rangle_N} \right)^{3/2} e^{-\{ \frac{3r^2}{2\langle \mathbf{r}^2 \rangle_N} + \frac{s}{\lambda_a} \}}$$

Green function for exactly N scatterings, valid in any regime!

P. B. Price et al., App. Op., **36** (1996).

Case (1b): Isotropic point source in Diffusive Regime

Derived by assuming large N limit of Eqs. (2) and (3).

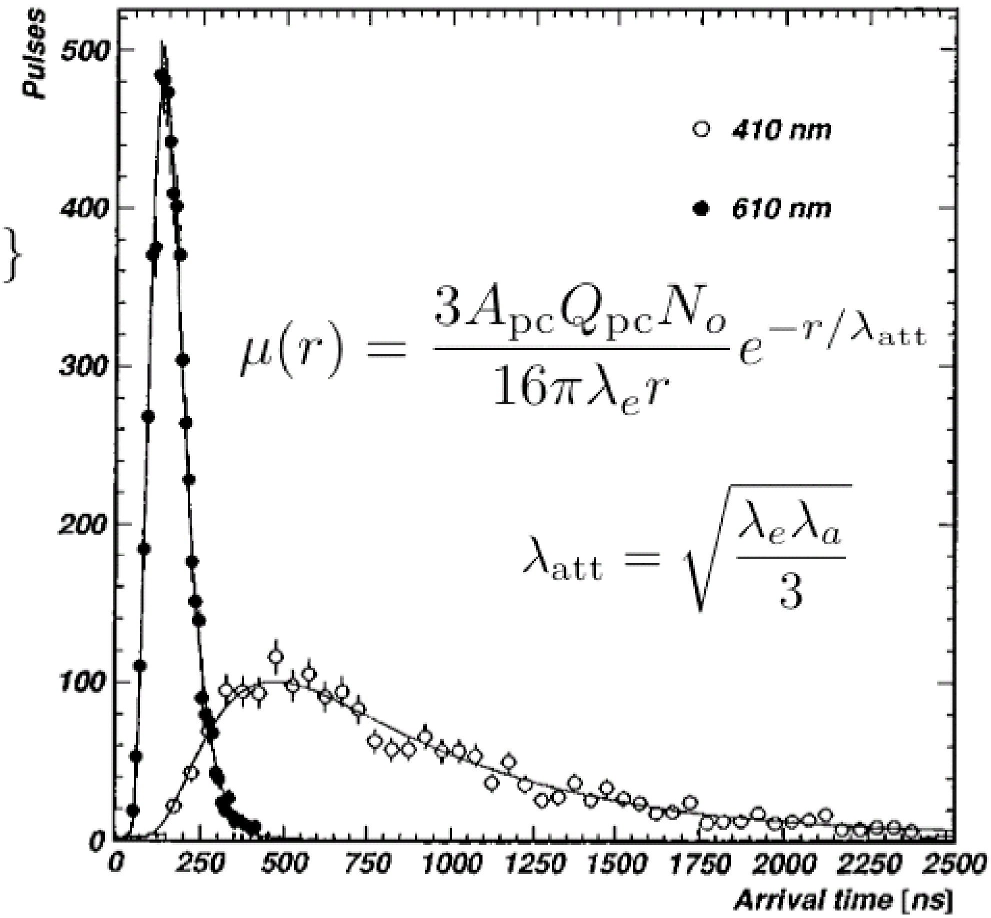
$$\langle \mathbf{r}^2 \rangle_N = 2N\lambda_e^2 \nu = 2\lambda_e s = 2\lambda_e c_i t$$

$$G(r,t) = \frac{1}{[4\pi D t]^{\frac{3}{2}}} \exp\{-\frac{r^2}{4Dt} - \frac{c_i}{\lambda_a} t\}$$

$$\langle t \rangle = \frac{r}{2c_i}\sqrt{\frac{3\lambda_a}{\lambda_e}}$$

$$\langle t^2 \rangle - \langle t \rangle^2 = \frac{r}{4c_i^2}\sqrt{\frac{3\lambda_a^3}{\lambda_e}}$$

- All desired quantities known.
- Non-causal.

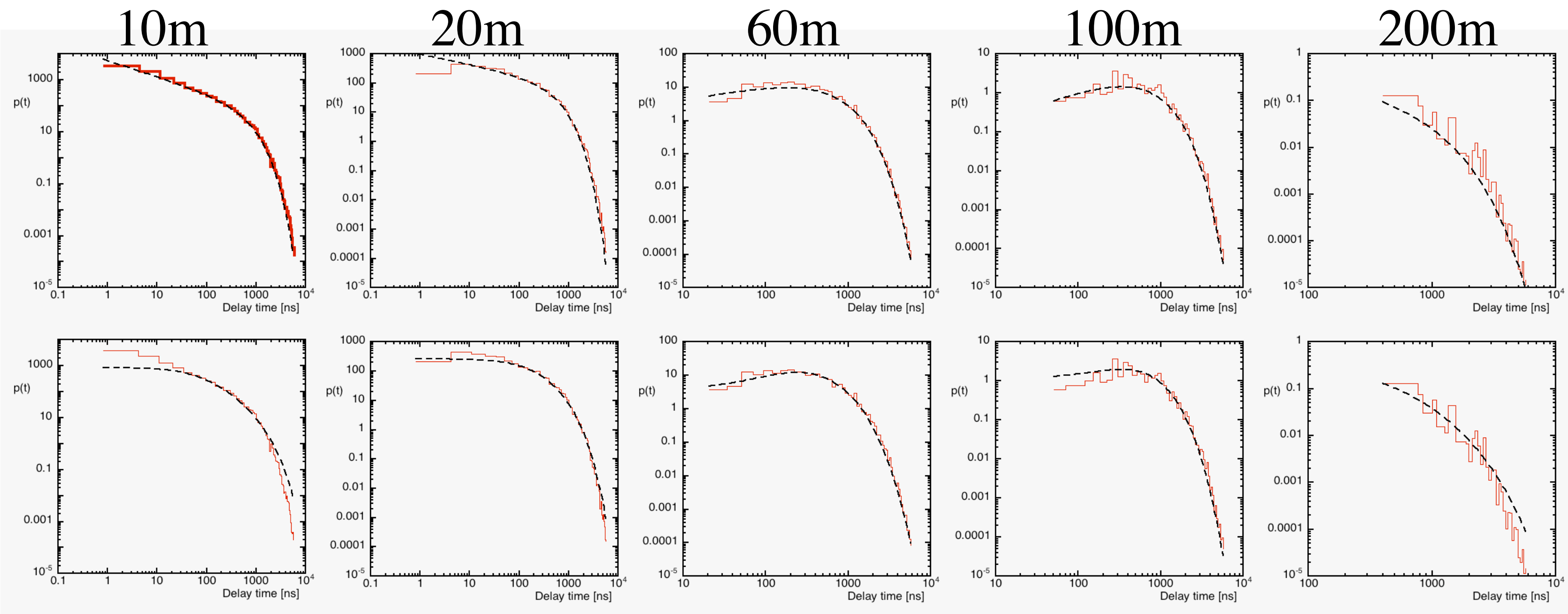


$$\mu(r) = \frac{3A_{\mathrm{pc}}Q_{\mathrm{pc}}N_o}{16\pi\lambda_e r} e^{-r/\lambda_{\mathrm{att}}}$$

$$\lambda_{\mathrm{att}} = \sqrt{\frac{\lambda_e \lambda_a}{3}}$$

P. B. Price et al., App. Op., **36** (1996).

Pandel and Diffusion Fits to Time Distributions from an Isotropic Point Source



Top Row: Pandel Fit

Bottom Row: Diffusion Fit

Time Distributions (Photonics)

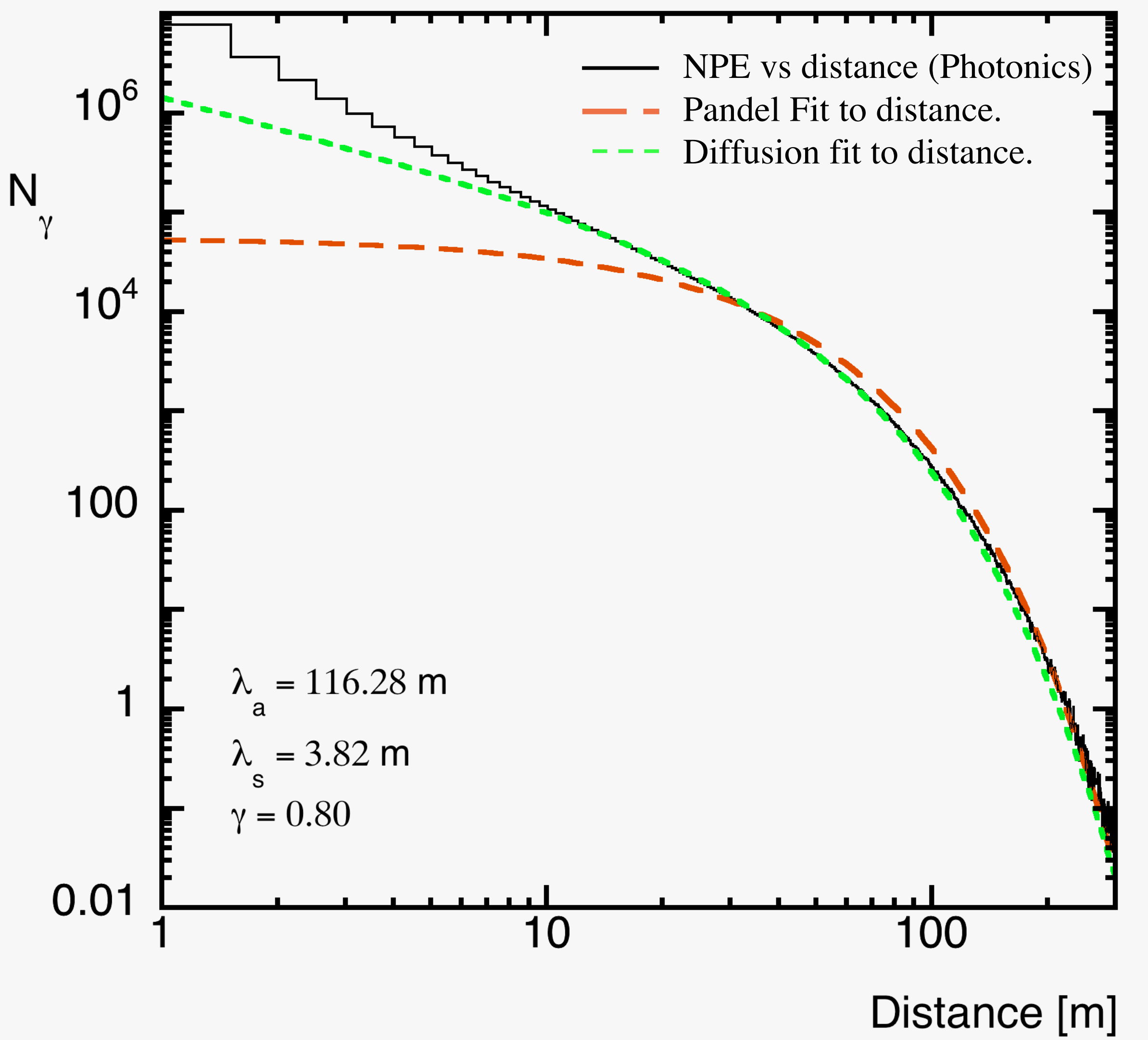
Fits

Conclusion: Gamma function gives the best fit to the time distribution!

R. Porrata, Berkeley, March 2005

Thanks to Michelangelo D'Agostino for generating photonics tables.

Pandel and Diffusion Fits vs Distance from an Isotropic Point Source



What gives?

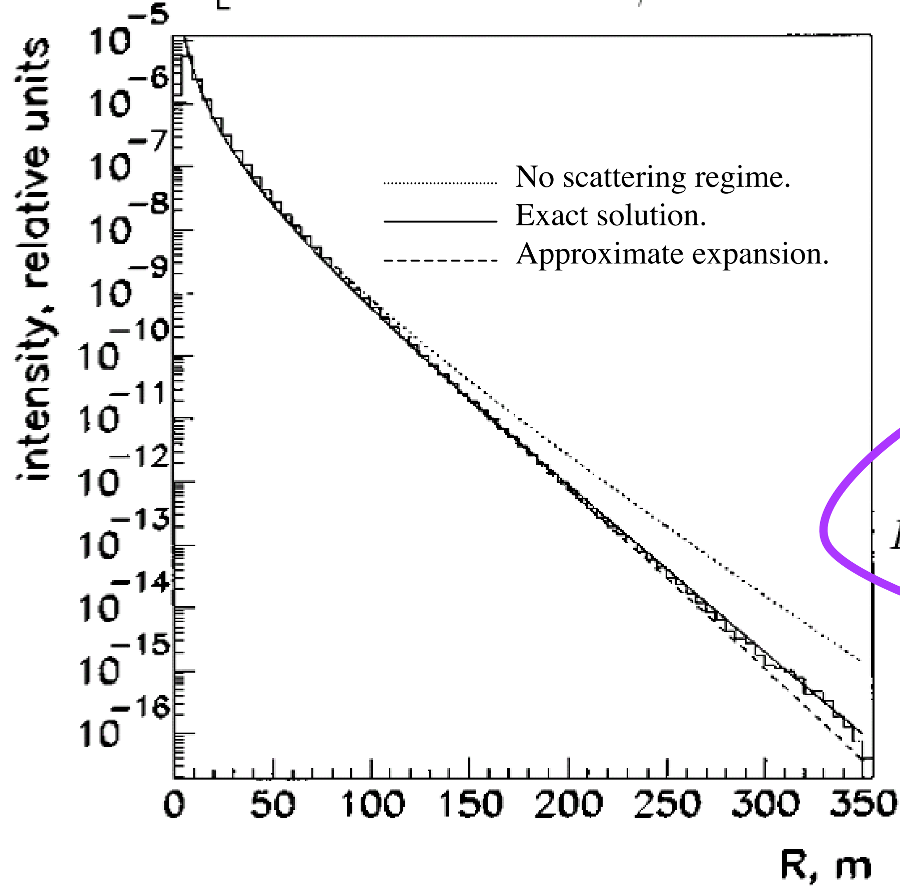
- Gauss' Law: Light should fall off as inverse square at short distances.
- Need to find another solution.

Thanks to Michelangelo D'Agostino for generating photonics tables.

Case (1d) : Isotropic point source in the Forward Scattering Regime

Transport DEQ in small angle approximation:

$$\left[(1-\frac{\psi^2}{2})\frac{\partial}{\partial z}-\frac{\psi}{r}\frac{\partial}{\partial\psi}+\kappa_1-\frac{1}{4}\sigma_1\langle\gamma^2\rangle\left(\frac{\partial^2}{\partial\psi^2}+\frac{1}{\psi}\frac{\partial}{\partial\psi}\right)\right]B(r,\psi)=\frac{I_o}{4\pi}\delta(\mathbf{r})$$



Solution found in an old Russian text:

$$B(r,\psi)=\frac{E}{\pi D}\exp(-\frac{\psi^2}{D}),$$

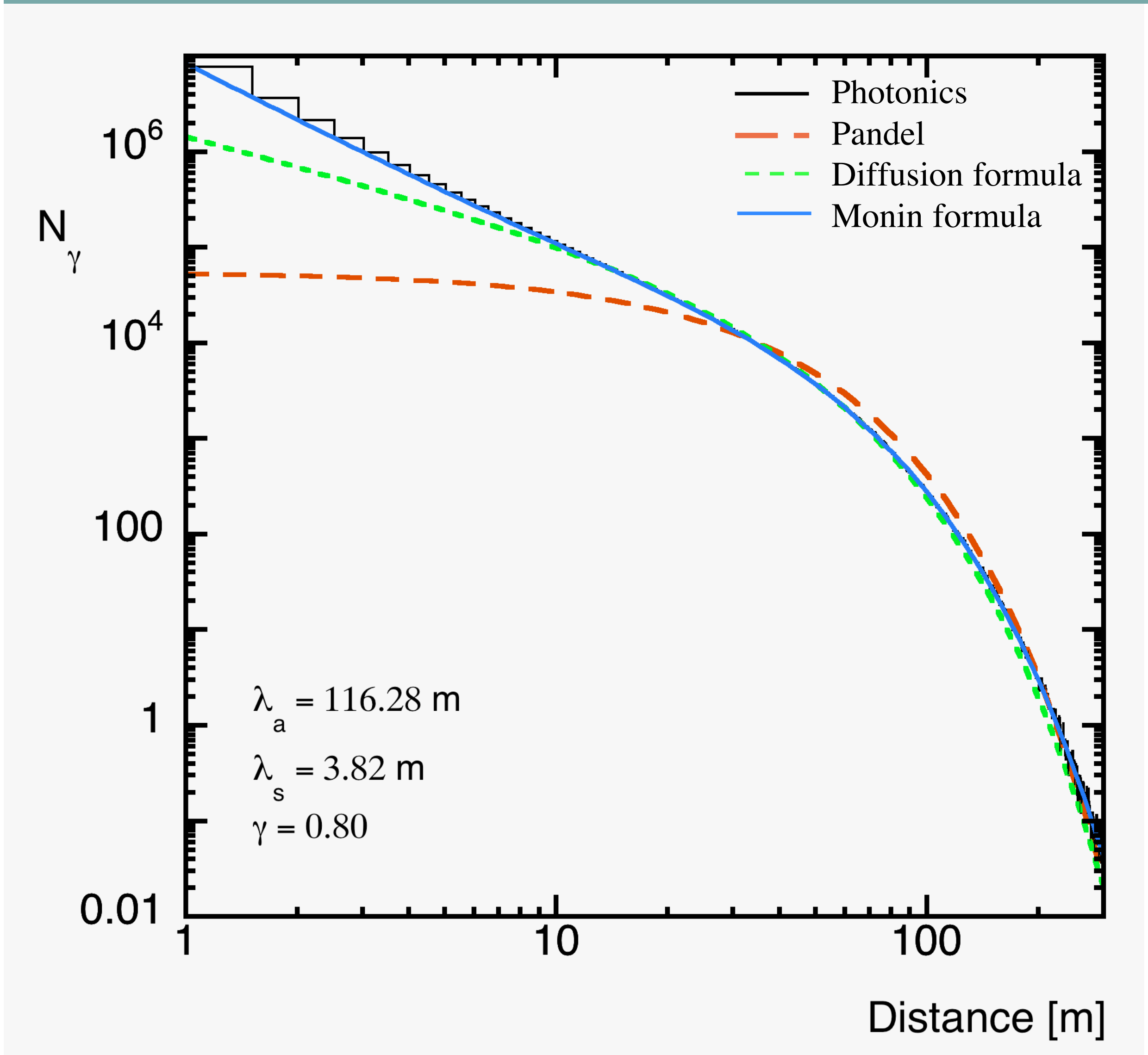
$$D(r)=D_\infty(\coth\rho-1/\rho),\qquad \rho=r\sqrt{\sigma_1\kappa_1\langle\gamma^2\rangle/2}$$

$$E=\frac{I_o e^{-\kappa_1 r}}{4\pi r^2}\left(\frac{\rho}{\sinh\rho}\right)\simeq\begin{cases}\dfrac{I_o e^{-\kappa_1 r}}{4\pi r^2} & \rho\ll 1,\\[2mm] \dfrac{(\alpha-\kappa_1)I_o}{2\pi r}e^{-\alpha r} & \rho\gg 1\end{cases}$$

Used by Baikal Collaboration to measure scattering and absorption.

A. S. Monin, Okeana Optika, **1,** (Nauka, Moscow (1983).

Pandel, Diffusion and Monin Fits to Distance from an Isotropic Point Source



Monin formula gives the best fit!

Thanks to Michelangelo D'Agostino for generating photonics tables.

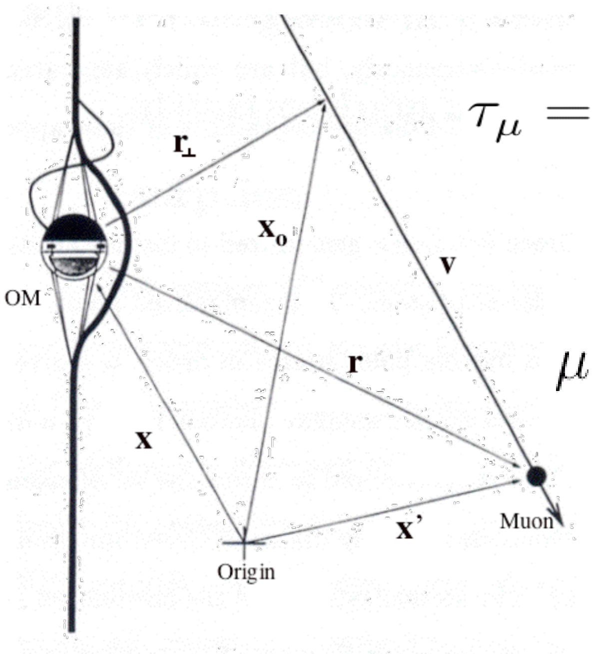
MOVING POINT SOURCES

(2)

Case (2b): Moving Point Sources in the Diffusive Regime

Everything is also known for a moving isotropic point source:

$$\frac{d\mu}{dt}(t, r_\perp, \lambda, E_\mu) = \frac{3c\ A_{\mathrm{pc}}\ Q_{\mathrm{pc}}}{16\pi\lambda_e r(t)} F(\lambda, E_\mu) \exp\{\frac{t}{\tau_\mu} - \frac{r(t)}{\lambda_\mu}\}$$



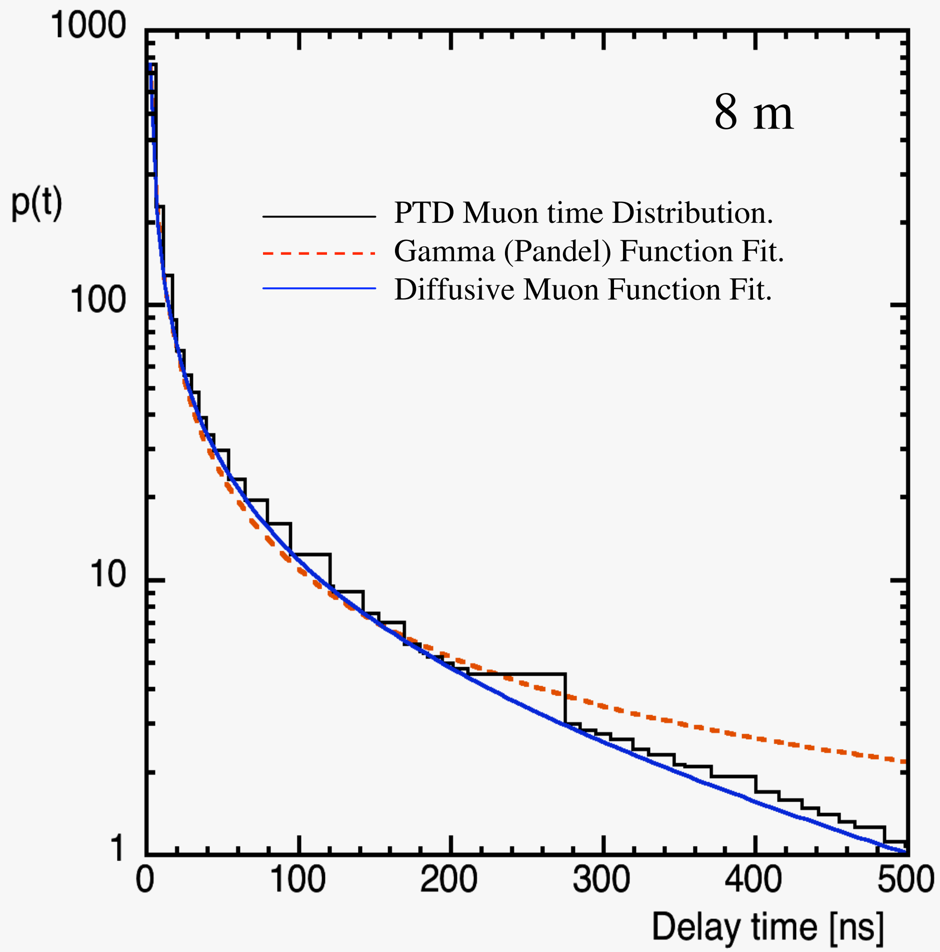
$$\tau_\mu = \frac{2\lambda_e}{3cn} \qquad \lambda_\mu = \frac{1}{\sqrt{(c\tau_\mu)^{-2} + \alpha^2}} \qquad r(t) = \sqrt{r_\perp^2 + c^2 t^2}$$

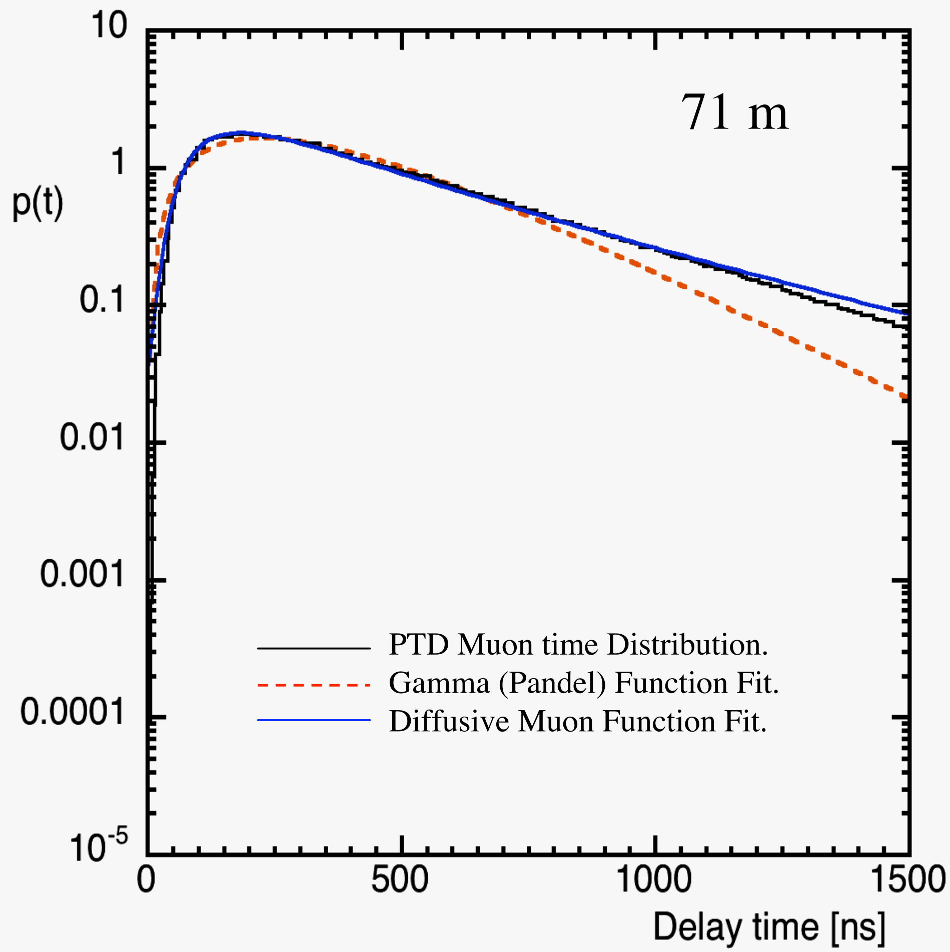
$$\mu(r_\perp, \lambda, E_\mu) = \frac{3\ A_{\mathrm{pc}}\ Q_{\mathrm{pc}}}{8\pi\lambda_e} F(\lambda, E_\mu) K_0(\alpha, r_\perp)$$

Until now, studies have only been performed by comparing simulation to data (Porrata 1997) or by reconstructing simulation (Wissing Bartol 2004). For a more direct comparison see next slide.

R. Porrata, App. (*Submitted Jan 2005*).

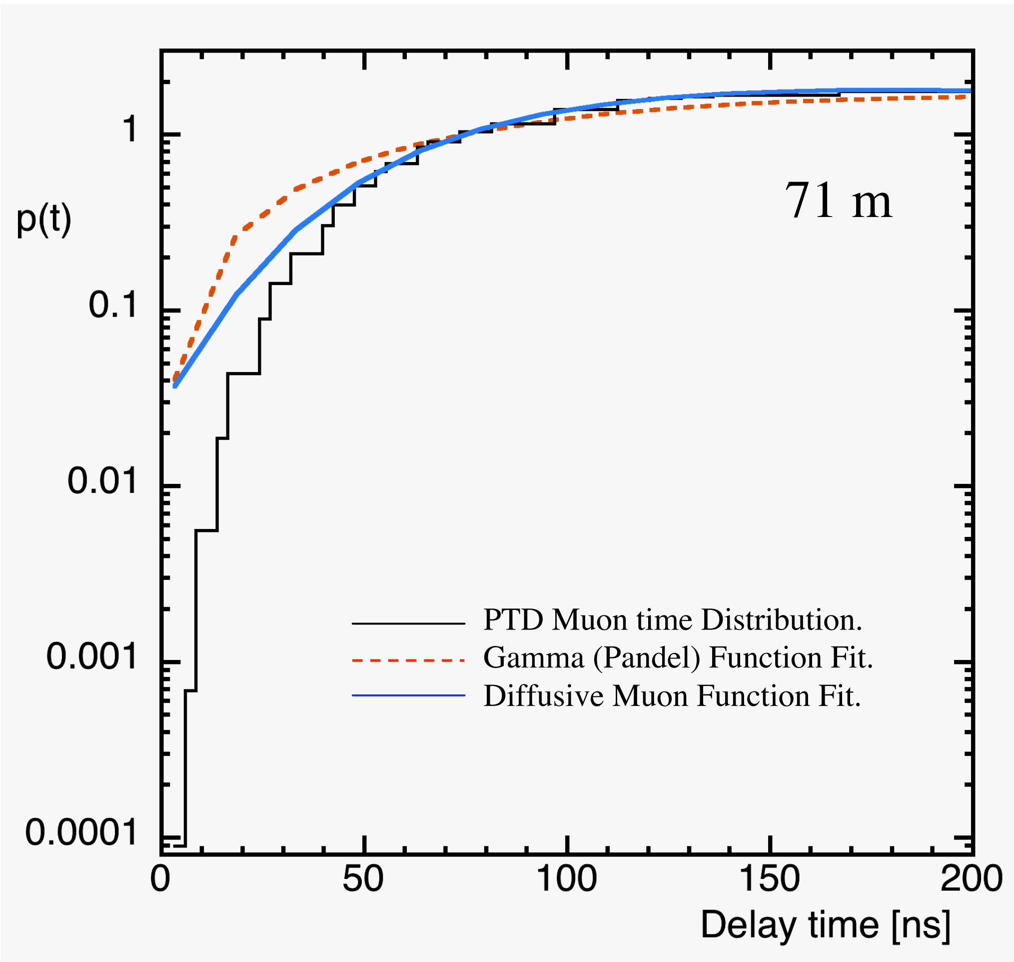
Pandel and Diffusive Muon Fits to Muon Time Distributions





- Pandel does **not** fit muon time distributions very well.
- Diffusive muon formula fits time distribution quite well.

Investigate Early Time Pandel and Diffusive Muon Fits to Muon Time Distributions



At early times:

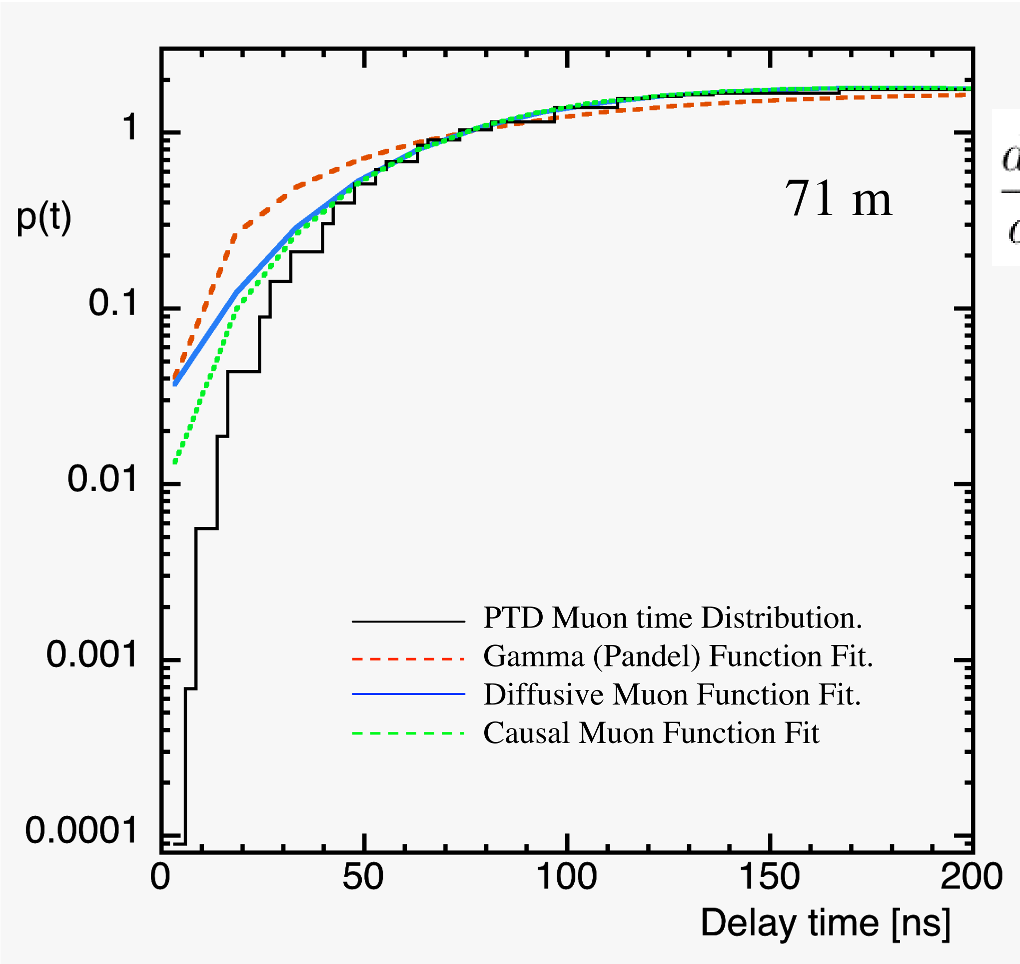
- Pandel fits poorly over first 70 ns but eventually makes it to origin.
- Moving diffusive point source does better, but it is *still* non-causal.

$$\frac{dn}{dt}(t, r_\perp) = \frac{I_o}{r(t)} \exp\{\frac{t}{\tau_\mu} - \frac{r(t)}{\lambda_\mu}\}$$

How to introduce causality?:

- Diffusive approx. finite at zero.
- Functional dependence must go to zero as ~*t*.
- Must approach diffusive result at large time.

Causal Modification of Diffusive Muon Formula for Muon Time Distributions



Causal diffusive muon formula:

$$\frac{dn}{dt}(t, r_\perp) = \frac{I_o}{r(t)} \sinh\{\frac{t}{\tau_\mu}\} \ \exp\{-\frac{r(t)}{\lambda_\mu}\}$$

- Best formula to use at very early times for large distances.
- Same good performance as non-causal diffusive muon formula for all other time ranges and distances.
- Becomes strongly negative in non-causal region (could be used as a signal to reco).

Analytic Fits to Muon Light Intensity versus Distance

Diffusive muon intensity formula has wrong dependence at short distances:

$$n(r_\perp/\lambda_{\rm att}) \sim K_0(r_\perp/\lambda_{\rm att}) \simeq \begin{cases} -\ln(r_\perp/\lambda_{\rm att}) & r_\perp/\lambda_{\rm att} \ll 1, \\ \dfrac{1}{\sqrt{2\pi r_\perp/\lambda_{\rm att}}}\exp\{-\dfrac{r_\perp}{\lambda_{\rm att}}\} & r_\perp/\lambda_{\rm att} \gg 1 \end{cases}$$

Again, need to use Gauss' law at short distances. Really want these limits:

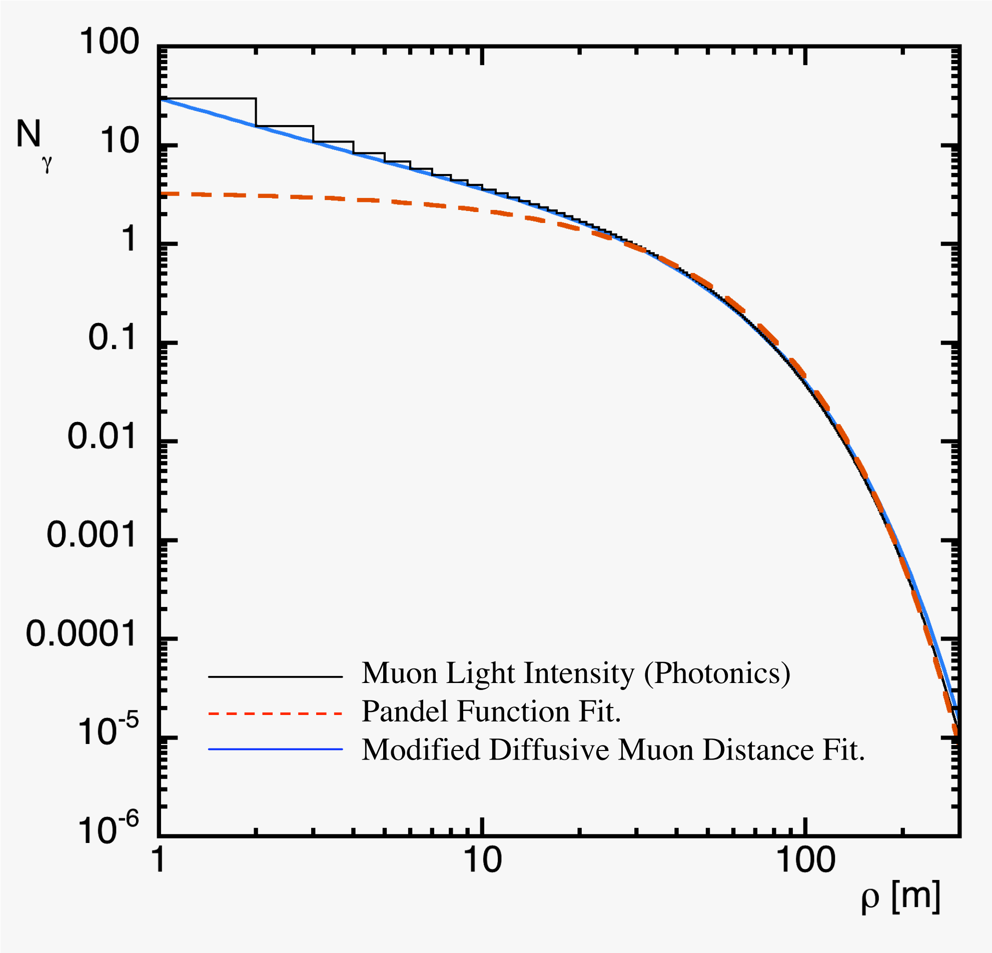
$$n(r_\perp/\lambda_{\rm att}) \simeq \begin{cases} \dfrac{1}{r_\perp/\lambda_{\rm att}} & r_\perp/\lambda_{\rm att} \ll 1, \\ \dfrac{1}{\sqrt{2\pi r_\perp/\lambda_{\rm att}}}\exp\{-\dfrac{r_\perp}{\lambda_{\rm att}}\} & r_\perp/\lambda_{\rm att} \gg 1 \end{cases}$$

Consider integrating Monin formula along a line. Difficult, but can use as a clue for how to do integral with the above limits in mind.

New Analytic Result for Muon Light Intensity

$$n(r_\perp) = \frac{I_o}{\sqrt{r_\perp/\lambda_\mu}\,\tanh(\sqrt{r_\perp/\lambda_\mu})}\exp\{-\frac{r_\perp}{\lambda_{\mathrm{att}}}\}$$

Modified Diffusive Muon Formula fits intensity versus distance for over 6.5 magnitude in number of photons.



Conclusions and "Reco"-mendations

What is OK to use:

- Gamma (Pandel) function well fits the **time** distribution from **point** sources. Use it for cascades.

That which is **not** OK to use:

- Gamma function does NOT provide a good fit to **muon** time distributions.
- Pandel normalization does NOT give the correct **NPE** with distance for either muons or cascades.

What one might consider using instead:

- Accurate fit of NPE with distance for a **point** source is given by Monin's formula. Why not use it?
- Accurate fit to NPE with distance for a muon given by new formula described here. Why not use it?
- Time distribution of photons from a muon is fit very well by non-causal and causal diffusive muon formulas given here.

Caveats:

- More work needed to characterize possible *distance* dependence of analytic causal and non-causal diffusive muon fits to **time** distributions.
- Of course this discussion has almost completely ignored angular distributions.